ISRAEL JOURNAL OF MATHEMATICS, Vol. 72, No. 3, 1990

GEOMETRICAL PROPERTIES OF SUBCLASSES OF COMPLEX L_1-PREDUALS

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ABSTRACT

We introduce the concept of a center of a set of complex functions. The concept is used to extend to complex spaces the max + min characterization of real G-spaces given by Lindenstrauss and Wulbert, to give geometrical and algebraic characterizations of complex C_σ-spaces and to characterize M-ideals.

Introduction

Lindenstrauss and Wulbert gave in [L–W] the following characterization of real G-spaces: A closed subspace V of a $C(X)$-space is a G-space if and only if, whenever $f, g \in V$, then $\max(f,g,0) + \min(f,g,0) \in V$.

The operation max + min has always been thought of as an extension of the lattice operations (i.e. max or min), and thus not valid in the complex case. But as demonstrated by Lima and Uttersrud [L–U], this is an incorrect way to view the operation max + min when G-spaces are concerned. Let instead c be defined by $c = \frac{1}{2}[\max(f,g,0) + \min(f,g,0)]$. Then $c(x)$ is, for each $x \in X$, the midpoint or center of the smallest closed interval containing $f(x)$, $g(x)$ and 0. This observation is in fact the clue to how the max + min operation should be extended to complex spaces.

Let $C_{\mathbf{C}}(X)$ be the space of all continuous complex valued functions on a compact Hausdorff space X. We define the center $c = c(f,g,0)$ of $f, g, 0 \in C_{\mathbf{C}}(X)$ to be the function $c: X \to \mathbf{C}$ where $c(x)$, for each $x \in X$, is the center of the smallest closed circular disk in $\mathbf{C}$ containing $f(x)$, $g(x)$ and 0. The function c becomes continuous, and obviously $c(f,g,0)$ and $\frac{1}{2}[\max(f,g,0) + \min(f,g,0)]$ will coincide when $f(x)$ and $g(x)$ are real.

Received April 5, 1990

Our main result is the following generalization of the Lindenstrauss–Wulbert characterization of real G-spaces: A closed subspace V of a $C_{\mathbf{C}}(X)$-space is a G-space if and only if, whenever $f,g \in V$, then $c(f,g,0) \in V$. As an easy consequence we get that the range of a contractive projection in a complex G-space is a G-space.

The M-ideals play an important role in our study. It turns out that a closed subspace J of a G-space V is an M-ideal if and only if, whenever $f \in J$ and $g \in V$, then $2c(g+f,g-f,0)-g \in J$. The M-ideals or, dually, the w^*-closed L-summands, define a topology on ext V_1^* (see [A–E]) that is called the Alfsen–Effros structure topology. Often it is more convenient to identify points $p,q \in$ ext V_1^* with $p=\lambda q$ for some $\lambda \in \mathbf{C}$, $|\lambda|=1$, and use the quotient space (ext $V_1^*)_\sigma$. As in the real case we get that a complex space V is a G-space if and only if the structure space (ext $V_1^*)_\sigma$ is Hausdorff.

Another max + min operation that is natural in this context is the operation $\max(f,g,1)+\min(f,g,-1)$. This operation has a complex extension too. Let $\mathbf{T}$ be the unit circle. We define $c=c(f,g,\mathbf{T})$ to be the function $c: X \to \mathbf{C}$ where $c(x)$, for each $x \in X$, is the center of the smallest closed disk containing $f(x)$, $g(x)$ and $\mathbf{T}$. We then get the following characterization of C_σ-spaces: A closed subspace V of a $C_{\mathbf{C}}(X)$-space is a C_σ-space if and only if, whenever $f,g \in V$, then $c(f,g,\mathbf{T}) \in V$. Again, as an easy consequence, we get that the range of a contractive projection in a complex C_σ-space is a C_σ-space.

The function $c(f,g,\mathbf{T})$ is $\mathbf{T}$-homogeneous since $\lambda c(f,g,\mathbf{T}) = c(\lambda f,\lambda g,\mathbf{T})$ for every $\lambda \in \mathbf{T}$. Thus, as a consequence of the Stone–Weierstrass Theorem, $c(f,g,\mathbf{T})$ can be approximated by $\mathbf{T}$-homogeneous polynomials in f, g, $\bar{f}$ and $\bar{g}$. We then obtain: $V \subseteq C_{\mathbf{C}}(X)$ is a C_σ-space if and only if $f^2\bar{f} \in V$ for each $f \in V$. Furthermore, a closed subspace J of a C_σ-space V is an M-ideal if and only if, whenever $f \in J$ and $g \in V$, then $f^2\bar{g} \in J$.

Notation

If V is a complex Banach space, then V^* is its dual space. The closed unit ball of V is written V_1 and $B(x,r)$ denotes the closed ball with center x and radius r. If K is a convex set then ext K is the set of extreme points of K. Especially, ext V_1^* is the set of extreme points of the closed unit ball of V^*.

In $\mathbf{C}$ let $\mathbf{T}$ denote the unit circle, i.e., $\mathbf{T}=\{\lambda \in \mathbf{C}: |\lambda|=1\}$. If $\lambda \in \mathbf{C}$ then $\bar{\lambda}$ is the complex conjugate of λ. $D(z,r)$ denotes the closed circular disk in $\mathbf{C}$ with center z and radius r. $C_{\mathbf{C}}(X)$ is the space of all continuous complex valued functions on a compact Hausdorff space X.

A closed subspace N of V is an L-summand if there is a subspace N' such that

V is the direct sum of N and N', and $\|x+y\| = \|x\| + \|y\|$ for every $x \in N$ and $y \in N'$. N' is called the complementary L-summand of N. For each L-summand N we have $\operatorname{ext} N_1 = N \cap \operatorname{ext} V_1$. A closed subspace J of V is called an M-ideal if the annihilator $N = J^\circ$ of J is an L-summand in V^*.

The w^*-closed L-summands of V^* define the closed sets of a topology on $\operatorname{ext} V_1^*$, i.e., $F \subseteq \operatorname{ext} V_1^*$ is closed if and only if $F = \operatorname{ext} N_1$ for some w^*-closed L-summand N of V^*. See [A–E]. In this topology $p, q \in \operatorname{ext} V_1^*$ can never be separated if $p = \lambda q$ for some $\lambda \in \mathbf{T}$. Hence it is often more convenient to identify such points and use the quotient space $(\operatorname{ext} V_1^*)_\sigma$.

1. The center of a bounded set in C

Let B be a bounded set in $\mathbf{C}$. Then there is a unique smallest closed circular disk containing B. We denote the center and the radius of this disk by $c(B)$ and $r(B)$. We shall say that $c(B)$ is the center and $r(B)$ the radius of B.

We will use this chapter to establish some useful facts about $c(B)$ and $r(B)$. These facts are based on elementary two-dimensional geometry. However, we have not been able to find any references that deal with these aspects of bounded sets of $\mathbf{C}$. Hence we sketch the proofs where we find it necessary, and we just state the results elsewhere.

Let $D(x,r)$ denote the closed circular disk in $\mathbf{C}$ with x as center and r as radius. We say that two sets B_1 and B_2 are δ-neighbours if there is, for every $z_1 \in B_1$, a $z_2 \in B_2$ such that $|z_1 - z_2| \le \delta$ and vice versa.

LEMMA 1.1. (i) *Let $\epsilon > 0$ and $R > 0$. Then there is a $\delta > 0$ such that $|c(B_1) - c(B_2)| \le \epsilon$ whenever B_1 and B_2 are δ-neighbours and $B_1, B_2 \subseteq D(0,R)$.*

(ii) *Let $\lambda \in \mathbf{C}$. Then $c(\lambda B) = \lambda c(B)$ and $r(\lambda B) = |\lambda| r(B)$.*

(iii) *Let $\bar{B} = \{\bar{b}\colon b \in B\}$. Then $c(\bar{B}) = \bar{c}(B)$ and $r(\bar{B}) = r(B)$.*

(iv) *$c(B + \lambda) = c(B) + \lambda$ and $r(B+\lambda) = r(B)$.*

PROOF. (i) Choose $\delta = \min(\epsilon^2/16R, \epsilon/2)$. Let B_1 and B_2 be δ-neighbours such that $B_1, B_2 \subseteq D(0,R)$. Let $r_1 = r(B_1)$, $r_2 = r(B_2)$ and $r = \max(r_1, r_2)$. Then $r \le R$. Let $c_1 = c(B_1)$ and $c_2 = c(B_2)$. We have $B_1 \subseteq D(c_2, r+\delta)$ and $B_2 \subseteq D(c_1, r+\delta)$, so $|c_1 - c_2| \le r + \delta$. If $r \le \delta$, then $|c_1 - c_2| \le 2\delta \le \epsilon$. Hence, we may assume $r > \delta$. Suppose $|c_1 - c_2| > \epsilon$. Let $2t = |c_1 - c_2|$. Since $B_1 \cup B_2 \subseteq D(c_1, r+\delta) \cap D(c_2, r+\delta)$, we have $B_1 \cup B_2 \subseteq D(a,s)$ where $2a = c_1 + c_2$ and $s^2 = (r+\delta)^2 - t^2$. Now $4t^2 > \epsilon^2 \ge 16\delta r$, and $t^2 > 4\delta r$. So $s < r$ since $s^2 < (r+\delta)^2 - 4\delta r = (r-\delta)^2$. We then have a contradiction to the minimality of either r_1 or r_2, depending on $r = r_1$ or $r = r_2$. Thus $|c_1 - c_2| \le \epsilon$.

(ii) There is nothing to prove if $\lambda = 0$. Let $\lambda \neq 0$. We always have $\lambda B \subseteq D(\lambda c(B), |\lambda| r(B))$. Thus $r(\lambda B) \leq |\lambda| r(B)$. Furthermore $r(B) = r(\lambda^{-1}\lambda B) \leq |\lambda^{-1}| r(\lambda B)$, and $|\lambda| r(B) \leq r(\lambda B)$. Since $c(\lambda B)$ is unique and $r(\lambda B) = |\lambda| r(B)$, we get $c(\lambda B) = \lambda c(B)$.

(iii) and (iv) are obvious.

Let B be finite or, more generally, a finite union of disks. Let $x_1, \ldots, x_n \in \mathbf{C}$ and $r_1, \ldots, r_n$ be non-negative numbers. We will write $c(x_1, \ldots, x_n)$ instead of $c(\{x_1, \ldots, x_n\})$ and we say that $c = c(x_1, \ldots, x_n)$ is the center of $x_1, \ldots, x_n$. In general $c_{\mathbf{r}} = c_{\mathbf{r}}(x_1, \ldots, x_n)$ is said to be the center of $x_1, \ldots, x_n$ with respect to $\mathbf{r} = (r_1, \ldots, r_n)$, defined by

$$(1.1) \qquad c_{\mathbf{r}}(x_1, \ldots, x_n) = c\left(\bigcup_{i=1}^{n} D(x_i, r_i)\right).$$

The radii $r(x_1, \ldots, x_n)$ and $r_{\mathbf{r}}(x_1, \ldots, x_n)$ are defined in the same way. We see that $c(x_1, \ldots, x_n)$ and $r(x_1, \ldots, x_n)$ are special cases of $c_{\mathbf{r}}$ and $r_{\mathbf{r}}$ with $\mathbf{r} = (0, \ldots, 0)$. From Lemma 1.1 we easily obtain:

LEMMA 1.2. *Let $x_1, \ldots, x_n, y, \lambda \in \mathbf{C}$ and $\mathbf{r} = (r_1, \ldots, r_n)$. Then we have:*

(i) *$\lambda c_{\mathbf{r}}(x_1, \ldots, x_n) = c_{|\lambda|\mathbf{r}}(\lambda x_1, \ldots, \lambda x_n)$ and $\lambda c(x_1, \ldots, x_n) = c(\lambda x_1, \ldots, \lambda x_n)$.*

(ii) *$c_r(x_1, \ldots, x_n) + y = c_r(x_1 + y, \ldots, x_n + y)$ and $c(x_1, \ldots, x_n) + y = c(x_1 + y, \ldots, x_n + y)$.*

(iii) *$c_r(\bar{x}_1, \ldots, \bar{x}_n) = \bar{c}_r(x_1, \ldots, x_n)$ and $c(\bar{x}_1, \ldots, \bar{x}_n) = \bar{c}(x_1, \ldots, x_n)$.*

(iv) *If $s_i = r_i + t \geq 0$ for $i = 1, \ldots, n$ and $s = (s_1, \ldots, s_n)$, then $c_s(x_1, \ldots, x_n) = c_{\mathbf{r}}(x_1, \ldots, x_n)$.*

We may also interpret $c(x_1, \ldots, x_n)$ (and $c_{\mathbf{r}}(x_1, \ldots, x_n)$) as the center of a largest disk. From the definition of $c(x_1, \ldots, x_n)$ and $r(x_1, \ldots, x_n)$ we get:

LEMMA 1.3. *Let $S = D(x_1, t) \cap \cdots \cap D(x_n, t)$, $c = c(x_1, \ldots, x_n)$ and $r = r(x_1, \ldots, x_n)$. Then $S \neq \emptyset$ if and only if $t \geq r$. If $S \neq \emptyset$ then c is the center of the largest circular disk contained in S, with radius equal to $t - r$.*

More generally, let $S = D(x_1, r_1) \cap \cdots \cap D(x_n, r_n)$, $t = \max(r_1, \ldots, r_n)$, $\mathbf{s} = (t - r_1, \ldots, t - r_n)$, $c_{\mathbf{s}} = c_{\mathbf{s}}(x_1, \ldots, x_n)$ and $r_{\mathbf{s}} = r_{\mathbf{s}}(x_1, \ldots, x_n)$. Then $S \neq \emptyset$ if and only if $t \geq r_{\mathbf{s}}$. If $S \neq \emptyset$ then $c_{\mathbf{s}}$ is the center of the largest circular disk contained in S, with radius equal to $t - r_{\mathbf{s}}$.

We will mainly be concerned about the case with just three elements, let us say x, y, and z. By Lemma 1.2, $c(x, y, z) = c(x - z, y - z, 0) + z$. Hence it is sufficient to study the case $z = 0$. Furthermore, let $\mathbf{r} = (r_1, r_2, r_3)$. If $r_1 = r_2 = r_3$ then

$c_{\mathbf{r}}(x,y,z) = c(x,y,z)$. If not, we may assume $r_1 \leq r_2 \leq r_3$ and $r_1 < r_3$. Then, by Lemma 1.2, $c_{\mathbf{r}}(x,y,z) = tc_{\mathbf{s}}(u,v,0)$ where $t = r_3 - r_1$, $s_2 = (r_2 - r_1)/t$, $\mathbf{s} = (0,s_2,1)$, $u = (x-z)/t$ and $v = (y-z)/t$. Again, the case $z = 0$ (with $r_1 = 0$ and $r_3 = 1$) is sufficient, i.e. $c_{\mathbf{r}}(x,y,0)$ with $\mathbf{r} = (0,r_2,1)$. In fact, it turns out that even $r_2 = 0$ is sufficient.

DEFINITION 1.4. Let $x,y \in \mathbf{C}$ and let $\mathbf{T}$ be the unit circle. We say that $c(x,y,\mathbf{T})$ is the center of x, y and $\mathbf{T}$, defined as the center of the smallest closed circular disk containing x, y and $\mathbf{T}$. Especially, $c(x,y,\mathbf{T}) = c(D(x,0) \cup D(y,0) \cup D(0,1)) = c_{\mathbf{r}}(x,y,0)$ with $\mathbf{r} = (0,0,1)$.

We see that if x,y are real then $c(x,y,0) = \frac{1}{2}[\max(x,y,0) + \min(x,y,0)]$ and $c(x,y,\mathbf{T}) = \frac{1}{2}[\max(x,y,1) + \min(x,y,-1)]$.

LEMMA 1.5. *Let* $x,y \in \mathbf{C}$, $c = c(x,y,0)$ *and* $r = r(x,y,0)$. *Then*

(i) $c = \frac{1}{2}y$, $r = \frac{1}{2}|y|$ *if* $|2x-y| \leq |y|$,

(ii) $c = \frac{1}{2}x$, $r = \frac{1}{2}|x|$ *if* $|2y-x| \leq |x|$,

(iii) $c = \frac{1}{2}(x+y)$, $r = \frac{1}{2}|x-y|$ *if* $|x+y| \leq |x-y|$ *and*

(iv) $c = \frac{1}{2}(x+y) + \frac{1}{2}i(y-x)\mathrm{Re}(\bar{x}y)/\mathrm{Im}(\bar{x}y)$, $r = \frac{1}{2}|xy(x-y)/\mathrm{Im}(\bar{x}y)|$ *else.*

PROOF. The triangle with x, y and 0 as vertices will either have just one angle bigger than or equal to $\pi/2$, or all angles less than $\pi/2$. In the first case c will be the midpoint of the edge opposite to the angle $\geq \pi/2$. In the second case c will coincide with the center of the circumscribed circle of the triangle.

(i) We see that the angle at x being bigger than or equal to $\pi/2$ is equivalent to $|2x-y| \leq |y|$. Thus $c = \frac{1}{2}y$ and $r = \frac{1}{2}|y|$.

(ii) and (iii) are obtained in a similar way.

(iv) Let L_1 and L_2 be the straight lines through $\frac{1}{2}(x+y)$ and $\frac{1}{2}x$ orthogonal to $y-x$ and x, respectively. Then c is the point of intersection of L_1 and L_2. We find c by solving the following equation with respect to s and t:

$$\frac{1}{2}(x+y) + ti(y-x) = \frac{1}{2}x + six.$$

By multiplying with $\bar{x}$ on each side and then equating the real parts, we get $\frac{1}{2}\mathrm{Re}(\bar{x}y) + t\,\mathrm{Re}(i\bar{x}y) = 0$. Thus $c = \frac{1}{2}(x+y) + \frac{1}{2}i(y-x)\mathrm{Re}(\bar{x}y)/\mathrm{Im}(\bar{x}y)$. This may be rewritten as $c = \frac{1}{2}ixy(\bar{y}-\bar{x})/\mathrm{Im}(\bar{x}y)$. Since $r = |c|$ the actual value of r is obtained.

LEMMA 1.6. *Let* $x,y \in \mathbf{C}$ *with* $|x|,|y| \leq 1$. *If* $z = 2c(y+x,y-x,0) - y$ *then* $|y+x-z| \leq 1$ and $|y-x-z| \leq 1$.

PROOF. We use Lemma 1.5 with $y + x$ and $y - x$ instead of y and x. Let $c = c(y+x, y-x, 0)$ and $z = 2c - y$. According to Lemma 1.5 we now have the four possibilities: (i) $|y - 2x| \leq |y|$, (ii) $|y + 2x| \leq |y|$, (iii) $|y| \leq |x|$ and (iv) else.

(i) Here $c = \frac{1}{2}(y + x)$, $z = x$, $y + x - z = y$ and $y - x - z = y - 2x$. So $|z| \leq 1$, $|y + x - z| \leq 1$ and $|y - x - z| \leq 1$.

(ii) and (iii) are obtained in a similar way.

(iv) Here $c = y - \frac{1}{2}x(y\bar{y} - x\bar{x})/\mathrm{Im}(\bar{x}y)$. The absolute value of $\mathrm{Im}(\bar{x}y)$ is twice the area of the triangle with x, y and 0 as vertices. Let h be the distance from 0 to the straight line through $y + x$ and $y - x$. Then $|\mathrm{Im}(\bar{x}y)| = h|x|$ and $|z - y| = (|y|^2 - |x|^2)/h$. The center c is on the straight line through y orthogonal to x. Hence $|y + x - z| \leq 1$ and $|y - x - z| \leq 1$ if and only if

$$|z - y| \leq (1 - |x|^2)^{1/2}. \tag{1.2}$$

Keep y fixed. Then the possible values of x are, since we now are in case (iv), all x such that $0 < |x| < |y| < \min(|y - 2x|, |y + 2x|)$. Choose s arbitrarily such that $0 < s < |y|$. Now it is sufficient to prove (1.2) for all x such that $|x| = s$. The smallest value h may achieve as x varies ($|x| = s$) is seen to be $(|y|^2 - s^2)^{1/2}$. Thus $|z - y| \leq (|y|^2 - s^2)^{1/2} \leq (1 - s^2)^{1/2} = (1 - |x|^2)^{1/2}$ since $|y| \leq 1$.

Let $|y|, |x_1|, |x_2| \leq 1$ and $z = 2c(y + x_1, y + x_2, 0) - y$. Then Lemma 1.6 tells that $|y + x_1 - z| \leq 1$ and $|y + x_2 - z| \leq 1$ if $x_1 = -x_2$. But it should be noted that we need not have $|y + x_1 - z| \leq 1$ and $|y + x_2 - z| \leq 1$ in general. Take, for instance, $y = i$, $x_1 = 1$ and $x_2 = 0$. Then $z = 2c(y + x_1, y + x_2, 0) - y = 2c(i + 1, i, 0) - i = 1$. Now $|y + x_1 - z| = |i| = 1$, but $|y + x_2 - z| = |i - 1| > 1$. This is in contrast to the real case. If y, x_1 and x_2 are real, then we always have $|y + x_1 - z| \leq 1$ and $|y + x_2 - z| \leq 1$.

We may regard $c(x,y,0)$ and $c(x,y,\mathbf{T})$ as functions from $\mathbf{C}^2 \to \mathbf{C}$, and by Lemma 1.1(i) they are both continuous. A function $f: \mathbf{C}^2 \to \mathbf{C}$ is said to be $\mathbf{T}$-homogeneous if $f(\lambda x, \lambda y) = \lambda f(x,y)$ for every $x, y \in \mathbf{C}$ and $\lambda \in \mathbf{T}$. By Lemma 1.2(i) both $c(x,y,0)$ and $c(x,y,\mathbf{T})$ are $\mathbf{T}$-homogeneous. A polynomial P in x, y, $\bar{x}$ and $\bar{y}$ is $\mathbf{T}$-homogeneous if and only if each term of P is of the form $\alpha x^m y^n \bar{x}^k \bar{y}^l$ where $m + n - (k + l) = 1$. This follows from the fact that if a and b are non-negative integers, then $\lambda^a \bar{\lambda}^b = \lambda$ for every $\lambda \in \mathbf{T}$ if and only if $a - b = 1$.

LEMMA 1.7. *Let $f(x,y) = c(x,y,0)$ or $f(x,y) = c(x,y,\mathbf{T})$. Let $\epsilon > 0$ and $R > 0$. Then there is a $\mathbf{T}$-homogeneous polynomial P in x, y, $\bar{x}$ and $\bar{y}$, with real coefficients, such that $|f(x,y) - P(x,y,\bar{x},\bar{y})| \leq \epsilon$ whenever $|x|, |y| \leq R$. The general term of P is of the form $\alpha x^m y^n \bar{x}^k \bar{y}^l$ where m, n, k and l are non-negative integers such* *that* $m + n - (k + l) = 1$, *and the two terms* $\alpha x^m y^n \bar{x}^k \bar{y}^l$ *and* $\beta x^n y^m \bar{x}^l \bar{y}^k$ *have* $\alpha = \beta$.

PROOF. By the Stone–Weierstrass Theorem there is a polynomial P_0 such that $|f(x,y) - P_0(x,y,\bar{x},\bar{y})| \le \epsilon$ whenever $|x|,|y| \le R$. Let N be the degree of P_0. Furthermore, let λ_j be a root of the equation $\lambda^j = -1$ for each $j = 1,\ldots,N+1$. We define P_j recursively by

$$P_j(x,y,\bar{x},\bar{y}) = \frac{1}{2}\,[P_{j-1}(x,y,\bar{x},\bar{y}) + \lambda_j^{-1}P_{j-1}(\lambda_j x,\lambda_j y,\overline{\lambda_j x},\overline{\lambda_j y})],$$

$$j = 1,2,\ldots,N+1.$$

Let $f_j(x,y) = f(x,y) - P_j(x,y,\bar{x},\bar{y})$. We claim that for each j we have $|f_j(x,y)| \le \epsilon$ whenever $|x|,|y| \le R$ and that P_j has no terms of the form $\alpha x^m y^n \bar{x}^k \bar{y}^l$ with $m + n - (k + l) - 1 \in \{\pm 1, \pm 2, \ldots, \pm j\}$. This may be proved by induction on j since for each $j = 1,2,\ldots,N+1$

$$|f_j(x,y)| \le \frac{1}{2}\,|f_{j-1}(x,y)| + \frac{1}{2}\,|\lambda_j^{-1}|\,|f_{j-1}(\lambda_j x,\lambda_j y)|, \qquad |x|,|y| \le R$$

and

$$1 + \lambda_j^{-1}\lambda_j^{m+n}\bar{\lambda}_j^{k+l} = 1 + \lambda_j^{\pm j} = 0 \qquad \text{if } m + n - (k + l) - 1 = \pm j.$$

The polynomial $P = P_{N+1}$ will meet the requirements of Lemma 1.7 except for realness and pairwise equality of the coefficients. But we may choose the coefficients real since $c(\bar{x},\bar{y},0) = \bar{c}(x,y,0)$ and $c(\bar{x},\bar{y},\mathbf{T}) = \bar{c}(x,y,\mathbf{T})$. Furthermore, $c(x,y,0) = c(y,x,0)$ and $c(x,y,\mathbf{T}) = c(y,x,\mathbf{T})$ give pairwise equality of the coefficients. For instance, the terms of degree 3 in such a polynomial will be $\alpha_1(x^2\bar{x} + y^2\bar{y})$, $\alpha_2(xy\bar{x} + xy\bar{y})$ and $\alpha_3(x^2\bar{y} + y^2\bar{x})$.

REMARK. There is a simpler proof of Lemma 1.7. Let $P = \int \lambda^{-1}P_0(\lambda x,\lambda y,\overline{\lambda x},\overline{\lambda y})\,d\lambda$ where the integration is with respect to the unit Haar measure on $\mathbf{T}$. Then P is $\mathbf{T}$-homogeneous, and P is a polynomial in x, y, $\bar{x}$ and $\bar{y}$ when P_0 is. Furthermore, $f(x,y) = \int \lambda^{-1} f(\lambda x,\lambda y)\,d\lambda$ since f is $\mathbf{T}$-homogeneous. Thus

$$|f(x,y) - P(x,y,\bar{x},\bar{y})| \le \int |\lambda^{-1}|\,|f(\lambda x,\lambda y) - P_0(\lambda x,\lambda y,\overline{\lambda x},\overline{\lambda y})|\,d\lambda \le \epsilon,$$

$$|x|,|y| \le R.$$

However, we prefer the first proof since it is more constructive.

PROBLEM. It would have been nice to have a concrete representation of the polynomials that Lemma 1.7 gives the existence of. We do believe that it is possible to construct such polynomials recursively, and that the odd Chebyshev polynomials might play a role.

LEMMA 1.8. *Let* $\epsilon > 0$. *Then there are an integer* k *and real coefficients* $\alpha_1,\ldots,\alpha_k$ *such that* $|x - (\alpha_1 x^2\bar{x} + \alpha_2 x^3\bar{x}^2 + \cdots + \alpha_k x^{k+1}\bar{x}^k)| \leq \epsilon$ *whenever* $|x| \leq 1$.

PROOF. This result must be well known, but we have not seen it stated anywhere. Hence we sketch a proof. Choose, for instance, a polynomial P (Lemma 1.7) such that $|c(x,y,0) - P(x,y,\bar{x},\bar{y})| \leq \epsilon/6$ whenever $|x|,|y| \leq 1$. Since $c(x,x,0) = \frac{1}{2}x = c(x,0,0)$ the coefficients $\alpha_1,\ldots,\alpha_k$ of the polynomial $Q(x,\bar{x}) = 4P(x,0,\bar{x},0) - 2P(x,x,\bar{x},\bar{x})$ will do the job. It is also possible to give a concrete representation of such coefficients. Choose n odd such that $1/n \leq \epsilon$, and let $2k+1 = n$. Let $P_{2k+1}(t)$ be the real Chebyshev polynomial of degree $2k+1$. Then $|P_{2k+1}(t)| \leq 1$ whenever $|t| \leq 1$. Now the coefficients $\alpha_1,\ldots,\alpha_k$ of the polynomial $Q(t) = [(2k+1)t - (-1)^k P_{2k+1}(t)]/(2k+1)$ will suit.

2. Complex G-spaces

A complex Banach space V is said to be a G-space if V is isometric to

(2.1)

$$W = \{f \in C_{\mathbf{C}}(X) : f(x_\alpha) = \lambda_\alpha f(y_\alpha),\ x_\alpha, y_\alpha \in X,\ \lambda_\alpha \in \mathbf{C},\ |\lambda_\alpha| \leq 1,\ \alpha \in A\}$$

for some compact Hausdorff space X and some set of indices A.

We need to define the concept of a center of a set of functions. Let $f_i \in C_{\mathbf{C}}(X)$, $i = 1,\ldots,n$. We define $c : X \to \mathbf{C}$ by $c(x) = c(f_1(x),\ldots,f_n(x))$ where $c(f_1(x),\ldots,f_n(x))$ is the center of $f_1(x),\ldots,f_n(x)$ as defined in Section 1. We shall say that c is the center of $f_1,\ldots,f_n$, and we denote it by $c = c(f_1,\ldots,f_n)$.

The concept of a center of $f_1,\ldots,f_n$ is an extension of the max + min operation to complex spaces. This is easily seen since, if f_i is real for each $i = 1,\ldots,n$, then

$$c(f_1,\ldots,f_n) = \frac{1}{2}\,[\max(f_1,\ldots,f_n) + \min(f_1,\ldots,f_n)].$$

LEMMA 2.1. *Let* $\lambda \in \mathbf{C}$, $x,y \in X$, $g,f_1,\ldots,f_n \in C_{\mathbf{C}}(X)$ *and* $c = c(f_1,\ldots,f_n)$. *Then*

(i) *c is continuous, i.e.,* $c \in C_{\mathbf{C}}(X)$.

(ii) *If* $f_i(x) = \lambda f_i(y)$ *for each* $i = 1,\ldots,n$ *then* $c(x) = \lambda c(y)$.

(iii) $c(\bar{f}_1,\ldots,\bar{f}_n) = \bar{c}(f_1,\ldots,f_n)$.

(iv) $c(f_1,\ldots,f_n) + g = c(f_1 + g,\ldots,f_n + g)$.

(v) *If* $\{f_\alpha\}$ *is bounded and equicontinuous instead of finite, then* (i)–(iv) *still hold.*

(vi) *If* W *is a complex G-space as in* (2.1) *and* $f_1,\ldots,f_n \in W$ *then* $c(f_1,\ldots,f_n) \in W$.

Proof. (i)–(v) follow from Lemma 1.1 and Lemma 1.2, and (vi) is a consequence of (2.1), (i) and (ii).

Definition 2.2. Let V be a complex Banach space and let $x,y \in V$. We say that $c(x,y,0) \in V$ if there is a $z \in V$ such that $p(z) = c(p(x),p(y),0)$ for every $p \in \operatorname{ext} V_1^*$. Such a z is uniquely determined and to simplify the notation we say $z = c(x,y,0)$.

Though $c(x,y,0)$ may fail to exist in V, it will always exist in a larger Banach space containing V (such as a $C(X)$ space that contains V).

Lemma 2.3. *Let V be a complex (or real) Banach space such that $c(x,y,0) \in V$ whenever $x,y \in V$. Let J be a closed subspace of V. Then the following statements are equivalent:*

(i) *J is an M-ideal.*

(ii) $J = \{x \in V: p(x) = 0, p \in F\}$ *for some* $F \subseteq w^*$-closure($\operatorname{ext} V_1^*$).

(iii) $2c(y + x,y - x,0) - y \in J$ *whenever* $x \in J$ *and* $y \in V$.

Proof. (i) ⇒ (ii). Let N be the annihilator of J. Then N is a w^*-closed L-summand in V^*, and $\operatorname{ext} N_1 = \operatorname{ext}(N \cap V_1^*)$. Let $F = \operatorname{ext} N_1$. Then obviously $J = \{x \in V: p(x) = 0, p \in F\}$.

(ii) ⇒ (iii). Let $x \in J$, $y \in V$ and $z = 2c(y + x,y - x,0) - y$. We have $p(z) = 2c(p(y + x),p(y - x),0) - p(y)$ for every $p \in \operatorname{ext} V_1^*$ and hence by continuity for every $p \in w^*$-closure($\operatorname{ext} V_1^*$). Let $p \in F$. Now $p(x) = 0$. So $p(z) = 2c(p(y),p(y),0) - p(y) = 2\frac{1}{2}p(y) - p(y) = 0$ and $z \in J$.

(iii) ⇒ (i). Let $x \in J$, $\|x\| \leq 1$ and $y \in V$, $\|y\| \leq 1$. By a theorem of Lima ([L$_1$, Theorem 6.15]) J is a semi M-ideal if there is a $z \in J$ such that $\|y + x - z\| \leq 1$ and $\|y - x - z\| \leq 1$. Let $z = 2c(y + x,y - x,0) - y$ and let $p \in \operatorname{ext} V_1^*$. Then $p(z) = 2c(p(y) + p(x),p(y) - p(x),0) - p(y)$. Now $|p(y)| \leq 1$, $|p(x)| \leq 1$ and then by Lemma 1.6 $|p(y) + p(x) - p(z)| \leq 1$, $|p(y) - p(x) - p(z)| \leq 1$. So $\|y + x - z\| \leq 1$ and $\|y - x - z\| \leq 1$ and J is a semi M-ideal.

A subspace J is said to have the 3 E.I.P. If $u,v \in J$ and $S = B(u,1) \cap B(v,1) \cap$

$B(0,1) \neq \emptyset$ implies that $J \cap B(u,1+\epsilon) \cap B(v,1+\epsilon) \cap B(0,1+\epsilon) \neq \emptyset$ for every $\epsilon > 0$. See the definition on page 104 and the note at the bottom of page 105 of [L_2]. A result of Lima ([L_2, Corollary 3.3]) says that a semi M-ideal with the 3 E.I.P. is already an M-ideal. The result is stated for real spaces only, but the proof also works in the complex case.

Let now $u, v \in J$ be such that $S = B(u,1) \cap B(v,1) \cap B(0,1) \neq \emptyset$. Let $y = (u+v)/2$, $x = (u-v)/2$ and $z = 2c(y+x, y-x, 0) - y$. Then by (iii) $z \in J$, and $c(u,v,0) \in J$ since $c(u,v,0) = (y+z)/2$. Let $p \in \text{ext } V_1^*$ and $S_p = D(p(u),1) \cap D(p(v),1) \cap D(0,1)$. Then $S_p \neq \emptyset$ since $S \neq \emptyset$. Hence, by Lemma 1.3, $c(p(u),p(v),0) \in S_p$ for each p and so $c(u,v,0) \in S$. We may now conclude that J has the 3 E.I.P. and hence is an M-ideal.

Remark. There is another way to prove that the semi M-ideal J is already an M-ideal. A complex Banach space V is said to be an $E(3)$-space if $D(p(u),1) \cap D(p(v),1) \cap D(0,1) \neq \emptyset$ for every $p \in \text{ext } V_1^*$ implies $S = B(u,1) \cap B(v,1) \cap B(0,1) \neq \emptyset$. By Lemma 1.3, $c(u,v,0) \in S$. Thus V is an $E(3)$-space. By a famous result of Lima [L–R], V is then an L_1-predual, and in L_1-preduals semi M-ideals and M-ideals do coincide [L_1].

We, however, prefer the proof using the 3 E.I.P. since it works both in the real and the complex case, while the $E(3)$-result holds for complex spaces only.

The w^*-closed L-summands of V^* (the annihilators of the M-ideals of V) define a topology on ext V_1^*. This topology is often called the Alfsen–Effros structure topology. See [A–E]. However, it does not separate dependent points of ext V_1^*. Instead we take the quotient space $(\text{ext } V_1^*)_\sigma$ obtained by identifying p and q if $p = \lambda q$ for some $\lambda \in \mathbf{T}$.

Theorem 2.4. *Let V be a complex (or real) Banach space. Then the following statements are equivalent:*

(i) *V is a G-space.*

(ii) *There is a compact Hausdorff space X and a subspace W of $C_{\mathbf{C}}(X)$, isometric to V, such that $c(f,g,0) \in W$ whenever $f,g \in W$.*

(iii) *If $x, y \in V$ then there is a $z \in V$ such that $p(z) = c(p(x),p(y),0)$ for every $p \in$ ext V_1^*.*

(iv) *The structure space $(\text{ext } V_1^*)_\sigma$ is Hausdorff.*

(v) *V is an L_1-predual, and w^*-closure(ext V_1^*) $\subseteq [0,1]$ ext V_1^*.*

Proof. (i) ⇒ (ii). The definition (2.1) of a G-space and Lemma 2.1(vi).

(ii) ⇒ (iii). Obvious.

(iii) ⇒ (iv). Let $q_1, q_2 \in \text{ext } V_1^*$ be independent. We need to find w^*-closed L-

summands N_1 and N_2 with $N_1 + N_2 = V^*$, $q_1 \in N_1$, $q_2 \in N_2$, $q_1 \notin N_2$ and $q_2 \notin N_1$. Choose $x,y \in V$ such that $q_1(x) = q_2(x) = q_1(y) = -q_2(y) = 1$. Let $t_p = \mathrm{Re}(\overline{p(x)}p(y))$, $p \in \mathrm{ext}\, V_1^*$. Let $F_1 = \{p \in \mathrm{ext}\, V_1^*: t_p \geq 0\}$ and $F_2 = \{p \in \mathrm{ext}\, V_1^*: t_p \leq 0\}$. Then $q_1 \in F_1$ and $q_2 \in F_2$. Let $J_i = \{v \in V: p(v) = 0, p \in F_i\}$, $i = 1,2$. By Lemma 2.3, J_i is an M-ideal and hence the annihilator $N_i = J_i^\circ$ an L-summand in V^*, $i = 1,2$. Now $N_1 + N_2 = V^*$ since $F_1 \cup F_2 = \mathrm{ext}\, V_1^*$, $q_1 \in N_1$ and $q_2 \in N_2$. Let $z_1, z_2 \in V$ be such that

$$p(z_1) = 2c(p(x),-p(y),0) - p(x) + p(y),$$

$$p(z_2) = 2c(p(x),p(y),0) - p(x) - p(y),$$

for every $p \in \mathrm{ext}\, V_1^*$. Let $p \in F_1$. Since $t_p \geq 0$ if and only if $|p(x) + p(y)| \geq |p(x) - p(y)|$, we have by Lemma 1.5(iii) that $c(p(x),-p(y),0) = \frac{1}{2}(p(x) - p(y))$. Hence $p(z_1) = 0$ and $z_1 \in J_1$. Now $q_2(z_1) = 2c(1,1,0) - 1 - 1 = -1$. Thus $q_2 \notin N_1$. In a similar way we find $z_2 \in J_2$ and $q_1(z_2) = -1$. So $q_1 \notin N_2$.

(iv) ⇒ (v). We shall prove that V is an L_1-predual by using [L_1, Theorem 5.8]. Let $q \in V^*$, $\|q\| = 1$. Suppose $P(q) = q$ or $P(q) = 0$ for every L-projection P in V^*. Let N be the smallest w^*-closed L-summand in V^* containing q. Suppose $\dim N \geq 2$. By assumption $(\mathrm{ext}\, V_1^*)_\sigma$ is Hausdorff. Hence we may choose w^*-closed L-summands N_1 and N_2, both different from N, such that $N_1 + N_2 = N$. Then $q \notin N_1$ and $q \notin N_2$. Let $L = N_1 + N'$ where N' is the complementary L-summand of N. Then $q \in L$, and $q \notin L'$ since $L' \subseteq N_2$. Let P be the L-projection associated with L. Then $P(q) \neq q$ and $P(q) \neq 0$. Hence we have got a contradiction to the assumption $\dim N \geq 2$. Then $\dim N = 1$, and $q \in \mathrm{ext}\, V_1^*$. Furthermore, if $q \in \mathrm{ext}\, V_1^*$ then $\mathrm{lin}(p)$ is an L-summand since $(\mathrm{ext}\, V_1^*)_\sigma$ is Hausdorff and then T_1. Hence by [L_1, Theorem 5.8] V^* is an L_1-space and V an L_1-predual.

Next let $q \in w^*$-closure$(\mathrm{ext}\, V_1^*)$. Then there is a net $\{p_\alpha\} \subseteq \mathrm{ext}\, V_1^*$ such that $p_\alpha \to q$ in the w^*-topology. Let N be the smallest w^*-closed L-summand containing q. Then by [A–E, Lemma 3.8] $p_\alpha \to p$ in the structure topology for every $p \in \mathrm{ext}(N \cap V_1^*)$. Hence N has to be 1-dimensional since $(\mathrm{ext}\, V_1^*)_\sigma$ is Hausdorff. So $q \in [0,1]\,\mathrm{ext}\, V_1^*$.

(v) ⇒ (i). Let $X = w^*$-closure$(\mathrm{ext}\, V_1^*)$ and let W be the space of all continuous, complex valued and T-homogeneous functions on X such that $f(x) = \|x\| f(x/\|x\|)$ for every $x \neq 0$. Then, as proved by Olsen [O], V is isometric to W and hence V is a G-space.

By Theorem 2.4 and by using the notation of Definition 2.2, we have that V is a G-space if and only if $c(x,y,0) \in V$ whenever $x, y \in V$. Now it is easy to prove results about contractive projections, quotient spaces with respect to M-ideals, intersection of M-ideals and about G-characters. A $q \in V^*$ is said to be a G-character if $q(c(x,y,0)) = c(q(x),q(y),0)$ for every $x,y \in V$. The concept of a G-character was introduced by Effros [E] for real G-spaces. Our definition is slightly different, but equivalent. Let $\mathbf{R}$ be the real numbers. Then $\mathbf{R}\,\mathrm{ext}\, V_1^* = \{tp\colon t \in \mathbf{R},\ p \in \mathrm{ext}\, V_1^*\}$.

Corollary 2.5. *Let V be a complex (or real) G-space. Then*:

(i) *The range of a contractive projection in V is a G-space.*

(ii) *If J is an M-ideal in V then the quotient space V/J is a G-space.*

(iii) *The intersection of any family of M-ideals in V is an M-ideal.*

(iv) *Every $q \in w^*$-closure(ext V_1^*) is a G-character. Moreover, q is a G-character if and only if $q \in \mathbf{R}$ ext V_1^*.*

Proof. (i) Let P be a contractive projection and $W = P(V)$. We will use arguments similar to the ones used in the proof of [L–W, Theorem 2(ii)]. Let $x,y \in W$. Then $c(x,y,0) \in V$. Let $q \in \mathrm{ext}\, W_1^*$, and F the set of all norm preserving extensions of q to V. Then F is a w^*-closed face of V_1^*. Let P^* be the dual projection. Then $P^*q \in F$. Now $p(c(x,y,0)) = c(p(x),p(y),0) = c(q(x),q(y),0)$ for every $p \in \mathrm{ext}\, F$ since $\mathrm{ext}\, F \subseteq \mathrm{ext}\, V_1^*$ and each $p \in \mathrm{ext}\, F$ is an extension of q. Now since $P^*q \in F$, we have $q(P(c(x,y,0)) = (P^*q)(c(x,y,0)) = c(q(x),q(y),0)$. Thus $P(c(x,y,0)) = c(x,y,0)$ and $c(x,y,0) \in W$. By Theorem 2.4, W is a G-space.

(ii) Let $\tilde{x}$ be the equivalence class of x with respect to J. Let N be the annihilator of J. Then we easily see that $c(\tilde{x},\tilde{y},0) = \tilde{c}(x,y,0)$ since $\mathrm{ext}(V/J)_1^* = \mathrm{ext}\, N_1 = N \cap \mathrm{ext}\, V_1^*$. So $c(\tilde{x},\tilde{y},0) \in V/J$ whenever $\tilde{x},\tilde{y} \in V/J$, and V/J is a G-space.

(iii) Let $\{J_\alpha\}$ be a family of M-ideals. Then by Theorem 2.4 and Lemma 2.3 we have $J_\alpha = \{x \in V\colon p(x) = 0,\ p \in F_\alpha\}$ and $\bigcap J_\alpha = \{x \in V\colon p(x) = 0,\ p \in \bigcup F_\alpha\}$. So $\bigcap J_\alpha$ is an M-ideal.

(iv) By continuity every $q \in w$-closure(ext V_1^*) is a G-character. We have $tc(p(x),p(y),0) = c(tp(x),tp(y),0)$ for every $t \in \mathbf{R}$ and $p \in \mathrm{ext}\, V_1^*$. So every $q \in \mathbf{R}$ ext V_1^* is a G-character. Suppose now q is a G-character, and let $J = \{x \in V\colon q(x) = 0\}$. Choose $x \in J$ and $y \in V$. Then $q(2c(y+x,y-x,0) - y) = 2c(q(y+x),q(y-x),0) - q(y) = 0$ since $q(x) = 0$. So $2c(y+x,y-x,0) - y \in J$. By Lemma 2.3, J is an M-ideal, and hence $\mathrm{lin}(q)$ is an L-summand. So $q \in \mathbf{R}$ ext V_1^*.

Remarks. In the real case, all of Theorem 2.4 and Corollary 2.5 are well known. The contributors have been Effros [E], Fakhoury [F], Lima [L_1], Lindenstrauss [LIN], Lindenstrauss and Wulbert [L–W], Taylor [T] and Uttersrud [U].

Complex G-spaces were introduced by Olsen [O] and (i) $\Leftrightarrow$ (v) of Theorem 2.4

was first proved by him. Rao [RAO] has proved (iv) ⇒ (v) and Corollary 2.5(iii). The rest of Theorem 2.4 and Corollary 2.5 seems to be new.

PROBLEMS. (1) Let P be a contractive projection in a G-space and let $W = P(V)$. Is $Pc(x,y,0) = c(Px,Py,0)$ for every $x,y \in V$? This is true if and only if $qP \in [0,1]$ ext V_1^* for every $q \in$ ext W_1^*.

(2) Property (iii) of Corollary 2.5 tells that a G-space has some rudiments of an algebraic structure. An interesting question is whether this property characterizes a G-space. To be more specific, suppose V is an L_1-predual such that the intersection of any family of M-ideals is an M-ideal. Is then V a G-space? It is true if V is separable. See [ROY], [RAO] or [LOU]. It is, however, not true if V is replaced by a space where $\ker(p)$ is an M-ideal in V for each $p \in$ ext V_1^*. (All L_1-preduals are of this type.) A counterexample (the disk algebra) is given in a forthcoming book by P. Harmand, D. Werner and W. Werner.

3. Complex C_σ-spaces

A compact Hausdorff space X is called a T_σ-space if there exists a map $\sigma : \mathbf{T} \times X \to X$ such that σ is continuous, $\sigma(1,x) = x$ and $\sigma(\alpha,\sigma(\beta,x)) = \sigma(\alpha\beta,x)$ for every $\alpha,\beta \in \mathbf{T}$ and $x \in X$. Let X be a T_σ-space. Then $f \in C_{\mathbf{C}}(X)$ is said to be σ-homogeneous if $f(\sigma(\alpha,x)) = \alpha f(x)$ for all $(\alpha,x) \in \mathbf{T} \times X$. The class of σ-homogeneous functions in $C_{\mathbf{C}}(X)$ is denoted by $C_\sigma(X)$. A complex Banach space V is said to be a C_σ-space if V is isometric to $C_\sigma(X)$ for some T_σ-space X.

We extend the definition of the center of a set of functions. Let $f_1,\ldots,f_n \in C_{\mathbf{C}}(X)$ and $\mathbf{r} = (r_1,\ldots,r_n)$. We define $c_{\mathbf{r}} : X \to \mathbf{C}$ by $c_{\mathbf{r}}(x) = c_{\mathbf{r}}(f_1(x),\ldots,f_n(x))$ (see (1.1)). We say that $c_{\mathbf{r}}(f_1,\ldots,f_n)$ is the center of $f_1,\ldots,f_n$ with respect to $\mathbf{r} = (r_1,\ldots,r_n)$. Especially, $c(f,g,\mathbf{T})$ is the center of $f,g,0$ with respect to $\mathbf{r} = (0,0,1)$. If all the functions are real, then

$$c_{\mathbf{r}}(f_1,\ldots,f_n) = \frac{1}{2}\,[\max(f_1 + r_1,\ldots,f_n + r_n) + \min(f_1 - r_1,\ldots,f_n - r_n)]$$

and

$$c(f,g,\mathbf{T}) = \frac{1}{2}\,[\max(f,g,1) + \min(f,g,-1)].$$

LEMMA 3.1. *Let* $\lambda \in \mathbf{C}$, $x,y \in X$, $f_1,\ldots,f_n \in C_{\mathbf{C}}(X)$, $\mathbf{r} = (r_1,\ldots,r_n)$ *and* $c_{\mathbf{r}} = c_{\mathbf{r}}(f_1,\ldots,f_n)$. *Then*

(i) $c_{\mathbf{r}}$ *is continuous, i.e.,* $c_{\mathbf{r}} \in C_{\mathbf{C}}(X)$.

(ii) *If* $f_i(x) = \lambda f_i(y)$, $|\lambda| = 1$, *for each* $i = 1, \dots, n$, *then* $c_{\mathbf{r}}(x) = \lambda c_{\mathbf{r}}(y)$.

(iii) *Let* V *be a* $C_\sigma(X)$*-space as defined above. If* $f_1, \dots, f_n \in V$, *then* $c_{\mathbf{r}}(f_1, \dots, f_n) \in V$. *Especially,* $c(f, g, \mathbf{T}) \in V$ *whenever* $f, g \in V$.

PROOF. (i) and (ii) follow from Lemma 1.1 and Lemma 1.2, and (iii) is a consequence of the definition of a $C_\sigma(X)$-space, of (i) and of (ii).

DEFINITION 3.2. Let V be a complex Banach space. Let $x, y, z \in V$. We say that $c(x, y, \mathbf{T}) \in V$ and $xy\bar{z} \in V$ if there are $u, v \in V$ such that $p(u) = c(p(x), p(y), \mathbf{T})$ (see Definition 1.4) and $p(v) = p(x)p(y)\overline{p(z)}$ for each $p \in \operatorname{ext} V_1^*$. Such points u and v are uniquely determined and to simplify the notation we will, where it is convenient, say $c(x, y, \mathbf{T}) = u$ and $xy\bar{z} = v$.

LEMMA 3.3. *Suppose* $x^2\bar{x} \in V$ *for every* $x \in V$. *If* $x, y, z \in V$ *and* m, n, k *and* l *are non-negative integers such that* $m + n - (k + l) = 1$, *then* $xy\bar{z} \in V$ *and* $x^m y^n \bar{x}^k \bar{y}^l \in V$.

PROOF. Let $x, y \in V$, and let $a = x + y$, $b = x - y$, $c = ix + y$ and $d = ix - y$. Then

$$x^2\bar{y} \in V \tag{3.1}$$

since $4x^2\bar{y} = a^2\bar{a} - b^2\bar{b} - c^2\bar{c} + d^2\bar{d}$. If $x, y, z \in V$, then $4xy\bar{z} = a^2\bar{z} - b^2\bar{z}$. Thus by (3.1)

$$xy\bar{z} \in V. \tag{3.2}$$

Let n be a positive integer. If $x_1, \dots, x_{n+1}, y_1, \dots, y_n \in V$, then the product $x_1 \cdots x_{n+1} \cdot \bar{y}_1 \cdots \bar{y}_n \in V$. By (3.2) this is easily proved by induction on n. Now the rest of Lemma 3.3 easily follows.

THEOREM 3.4. *Let* V *be a complex (or real) Banach space. Then the following statements are equivalent:*

(i) V *is a* C_σ*-space.*

(ii) V *is isometric to* $W = \{f \in C_{\mathbf{C}}(X) : f(x_\alpha) = \lambda_\alpha f(y_\alpha),\ x_\alpha, y_\alpha \in X,\ \lambda_\alpha \in \mathbf{T} \cup \{0\},\ \alpha \in A\}$ *for some compact Hausdorff space* X *and set of indices* A.

(iii) *There is a subspace* W, *isometric to* V, *of some* $C_{\mathbf{C}}(X)$*-space such that* $f^2\bar{f} \in W$ *for each* $f \in W$.

(iv) *There is a subspace* W, *isometric to* V, *of some* $C_{\mathbf{C}}(X)$*-space such that* $c(f, g, \mathbf{T}) \in W$ *whenever* $f, g \in W$.

(v) *If* $x \in V$, *then there is a* $z \in V$ *such that* $p(z) = p^2(x)\overline{p(x)}$ *for each* $p \in \operatorname{ext} V_1^*$.

(vi) *If* $x,y \in V$, *then there is a* $z \in V$ *such that* $p(z) = c(p(x),p(y),\mathbf{T})$ *for each* $p \in \mathrm{ext}\, V_1^*$.

(vii) V *is an* L_1*-predual, and* w^*-closure(ext V_1^*) $\subseteq \{0\} \cup \mathrm{ext}\, V_1^*$.

In the real case, property (iv) (and (vi)) reminds one of the Lindenstrauss–Wulbert characterization of G-spaces. Remember that if $C(X)$ is real, then $V \subseteq C(X)$ is a G-space if and only if, whenever $f,g \in V$, then $\max(f,g,0) + \min(f,g,0) \in V$. For C_σ-spaces we get:

COROLLARY 3.5. *Let* V *be a real Banach space. Then the following statements are equivalent*:

(i) V *is a* C_σ*-space.*

(ii) *There is a subspace* W, *isometric to* V, *of some real* $C(X)$*-space such that, whenever* $f,g \in W$, *then* $\max(f,g,1) + \min(f,g,-1) \in W$.

(iii) *If* $x,y \in V$, *then there is a* $z \in V$ *such that* $p(z) = \max(p(x),p(y),1) + \min(p(x),p(y),-1)$ *for every* $p \in \mathrm{ext}\, V_1^*$.

PROOF OF THEOREM 3.4. (i) ⇒ (ii). Follows from the definition of a C_σ-space.

(ii) ⇒ (iii). Let $\lambda \in \{0\} \cup \mathbf{T}$ and suppose $f(x) = \lambda f(y)$. Then $(f^2\bar{f})(x) = f^2(x)\overline{f(x)} = \lambda^2\bar{\lambda} f^2(y)\overline{f(y)} = \lambda(f^2\bar{f})(y)$.

(ii) ⇒ (iv). Lemma 3.1.

(iii) ⇒ (v) and (iv) ⇒ (vi) are obvious.

(v) ⇒ (vi). Let $x,y \in V$ and $R = \max(\|x\|,\|y\|)$. Then $|p(x)|,|p(y)| \leq R$ for every $p \in \mathrm{ext}\, V_1^*$. Let $n > 0$. By Lemma 1.7 there is a $\mathbf{T}$-homogeneous polynomial Q_n such that

$$|c(p(x),p(y),\mathbf{T}) - Q_n(p(x),p(y),\overline{p(x)},\overline{p(y)})| \leq 1/n \qquad \text{for every } p \in \mathrm{ext}\, V_1^*,$$

and by Lemma 3.3, $z_n = Q_n(x,y,\bar{x},\bar{y}) \in V$. Let $\epsilon > 0$ and choose $N \geq 2/\epsilon$. Let $n,m \geq N$. Then $\|z_n - z_m\| \leq \epsilon$. So $\{z_n\}$ is a Cauchy sequence. Let $z_n \to z$. Then obviously $z = c(x,y,\mathbf{T})$. (The same argument also implies that $c(x,y,0) \in V$.)

(vi) ⇒ (vii). Let $x,y \in V$. We shall prove $c(x,y,0) \in V$. Choose $u_n \in V$, $n = 1,2,\ldots$, such that $p(u_n) = c(p(nx),p(ny),\mathbf{T})$ for every $p \in \mathrm{ext}\, V_1^*$. Let $\epsilon > 0$ and $R = \max(\|x\|,\|y\|,1)$. Choose δ as in Lemma 1.1(i), and let $N \geq 1/\delta$. Let $B = \{p(x),p(y),0\}$, $B_n = B \cup (1/n)\mathbf{T}$ and $z_n = (1/n)u_n$. By Lemma 1.1(i), if $n \geq N$ and $p \in \mathrm{ext}\, V_1^*$ then $|p(z_n) - c(p(x),p(y),0)| = |c(B_n) - c(B)| \leq \epsilon$. So $\{z_n\}$ is a Cauchy sequence. Let $z_n \to z$. Then obviously $z = c(x,y,0)$. By Theorem 2.4, V is a G-space, and hence an L_1-predual.

Let now $q \in w^*$-closure(ext V_1^*). Then, by Theorem 2.4(v), $q = tp$, $t \in [0,1]$ and $p \in \mathrm{ext}\, V_1^*$. Suppose $t > 0$. Choose $y \in V$ such that $p(y) = 1$. Let

$$x = \frac{2}{t} y \quad \text{and} \quad z = c(x,x,\mathbf{T}).$$

By continuity, $q(z) = c(q(x), q(x), \mathbf{T})$. Now

$$p(z) = c(p(x), p(x), \mathbf{T}) = c\left(\frac{2}{t}, \frac{2}{t}, \mathbf{T}\right) = \frac{1}{t} - \frac{1}{2}$$

and

$$q(z) = c(tp(x), tp(x), \mathbf{T}) = c(2,2,\mathbf{T}) = \frac{1}{2}.$$

But $q(z) = tp(z)$, hence $\frac{1}{2} = 1 - t/2$ and $t = 1$. Thus $q \in \{0\} \cup \operatorname{ext} V_1^*$.

(vii) ⇒ (i). Let $X = \operatorname{ext} V_1^* \cup \{0\}$ and let $\sigma : \mathbf{T} \times X \to X$ be defined by $\sigma(\lambda, x) = \lambda x$. Then, as proved by Olsen [O], V is isometric to $C_\sigma(X)$.

We use now the notation of Definition 3.2:

THEOREM 3.6. *Let V be a complex (or real) C_σ-space and J a closed subspace of V. Then the following statements are equivalent:*

(i) *J is an M-ideal.*
(ii) *$x^2\bar{y} \in J$ whenever $x \in J$ and $y \in V$.*
(iii) *$xy\bar{z} \in J$ whenever $x, y, z \in V$ and at least one of them are in J.*

PROOF. (i) ⇒ (ii). A C_σ-space is a G-space. Hence by Lemma 2.3, $J = \{x \in V: p(x) = 0, p \in F\}$ for some $F \subseteq \operatorname{ext} V_1^*$. Let $p \in F$. Then $p(x^2\bar{y}) = p^2(x)\overline{p(y)} = 0$ since $p(x) = 0$.

(ii) ⇒ (iii). We may assume $\|x\|, \|y\|, \|z\| \leq 1$. Let $\epsilon > 0$. If $\lambda \in \mathbf{C}$, then by Lemma 1.8 we may choose a polynomial P in λ and $\bar{\lambda}$ such that

(3.3) $$|\lambda - P(\lambda, \bar{\lambda})| \leq \epsilon, \quad |\lambda| \leq 1 \quad \text{and}$$

(3.4) $$\alpha\lambda^{k+1}\bar{\lambda}^k, \quad k \geq 1 \quad \text{is the general term.}$$

Suppose $x \in J$. Then $P(x,\bar{x})y\bar{z} \in J$ since by (3.4) $\alpha x^{k+1}\bar{x}^k y\bar{z} = \alpha x^2\bar{u}$ where $u = x^k z \bar{x}^{k-1}\bar{y}$ and by Lemma 3.3 $u \in V$. Furthermore, by (3.3), $\|xy\bar{z} - P(x,\bar{x})y\bar{z}\| \leq \epsilon$. So $xy\bar{z} \in J$ since $\epsilon > 0$ is arbitrarily chosen and J is closed. If $y \in J$ then we get $xy\bar{z} \in J$ in the same way. Let instead $z \in J$. Then $\|xy\bar{z} - xyP(\bar{z}, z)\| \leq \epsilon$, and $\alpha xy\bar{z}^{k+1}z^k = \alpha xz\bar{u} \in J$ since $z \in J$ and $u = z^{k+1}\bar{y}\bar{z}^{k-1} \in V$. So $xy\bar{z} \in J$.

(iii) ⇒ (i). Let $x \in J$, $y \in V$ and $z = 2c(y + x, y - x, 0) - y$. Then by Theorem 2.4, $z \in V$ since V is a C_σ-space and hence a G-space. Let $\epsilon > 0$ and $R =$

$\max(\|y+x\|,\|y-x\|.)$ Since $\frac{1}{2}(z+y)=c(y+x,y-x,0)$, we may, by Lemma 1.7, choose a **T**-homogeneous polynomial P such that $\|\frac{1}{2}(z+y)-P(y+x,y-x,\overline{y+x},\overline{y-x})\|\leq\epsilon$. Then $P(y+x,y-x,\overline{y+x},\overline{y-x})-P(y,y,\bar{y},\bar{y})$ will be a polynomial $Q(x,y,\bar{x},\bar{y})$ where each term will be of the form $xu\bar{v}$ or $uv\bar{x}$, $u,v\in V$. So by (iii) $Q(x,y,\bar{x},\bar{y})\in J$. Furthermore

$$\|z-2Q(x,y,\bar{x},\bar{y})\|\leq\|x+y-2P(y+x,y-x,\overline{y+x},\overline{y-x})\|$$
$$+\|2P(y,y,\bar{y},\bar{y})-y\|\leq 2\epsilon+2\epsilon=4\epsilon$$

since $\|y\|\leq R$ and $y=2c(y,y,0)$. Thus $z\in J$ since $\epsilon>0$ is arbitrarily chosen and J is closed. By Lemma 2.3 J is an M-ideal.

The different characterizations of a C_σ-space and of an M-ideal may now be used to prove results about contractive projections, quotient spaces and C_σ-characters. We could, as in the case of a G-space, define a C_σ-character by using the center $c(x,y,\mathbf{T})$, i.e. $q\in V^*$ is a C_σ-character if $q(c(x,y,\mathbf{T}))=c(q(x),q(y),\mathbf{T})$ for every $x,y\in V$. But the concept of a C_σ-character was introduced for real C_σ-spaces by Effros [E], and we follow his definition: A $q\in V^*$ is said to be a C_σ-character if and only if $q(xy\bar{z})=q(x)q(y)\overline{q(z)}$ for every $x,y,z\in V$.

Corollary 3.7. *Let V be a complex (or real) C_σ-space. Then:*

(i) *The range of a contractive projection in V is a C_σ-space.*

(ii) *If J is an M-ideal in V, then the quotient space V/J is a C_σ-space.*

(iii) *Let $q\in V^*$. Then the following statements are equivalent:*

(1) *q is a C_σ-character.*

(2) *$q(x^2\bar{x})=q^2(x)\bar{q}(x)$ for every $x\in V$.*

(3) *$q(c(x,y,\mathbf{T}))=c(q(x),q(y),\mathbf{T})$ for every $x,y\in V$.*

(4) *$q\in\{0\}\cup\operatorname{ext}V_1^*$.*

Proof. (i) Let W and P be as in the proof of Corollary 2.5(i). Let $x\in W$. Then, by using the same type of arguments, we find $x^2\bar{x}\in W$. Or if $x,y\in W$, we find $c(x,y,\mathbf{T})\in W$. So by Theorem 3.4, $W=P(V)$ is a C_σ-space.

(ii) Let, as in the proof of Corollary 2.5(ii), $\tilde{x}$ be the equivalence class of x with respect to J. Then, in the same way, we find $\tilde{c}(x,y,\mathbf{T})=c(\tilde{x},\tilde{y},\mathbf{T})$, or $\tilde{z}=\tilde{x}^2\bar{\tilde{x}}$ where $z=x^2\bar{x}$. So by Theorem 3.4, V/J is a C_σ-space.

(iii) (1) ⇒ (2), (4) ⇒ (1) and (4) ⇒ (3) are obvious.

(2) ⇒ (4). Let $J=\{x\in V\colon q(x)=0\}$. Let $x\in J$ and $y\in V$. Then $4x^2\bar{y}=a^2\bar{a}-b^2\bar{b}-c^2\bar{c}+d^2\bar{d}$ where $a=x+y$, $b=x-y$, $c=ix+y$ and $d=ix-y$. Then $q(x^2\bar{y})=0$ since $q(x)=0$. So by Theorem 3.6, J is an M-ideal and then $\operatorname{lin}(q)$ an L-summand. Hence $q=tp$, $t\geq 0$ and $p\in\operatorname{ext}V_1^*$. Choose $y\in V$ such

that $p(y) = 1$. Then $q(y^2\bar{y}) = q^2(y)\overline{q(y)} = t^3p^2(y)\overline{p(y)} = t^3 = tp(y^2\bar{y}) = t$. So $t = 0$ or 1.

(3) ⇒ (4). In the same way as in the proof of Theorem 3.4(vi) ⇒ (vii), we find that $q(c(x,y,0) = c(q(x),q(y),0)$ for every $x, y \in V$. So q is a G-character, and then $q = tp$, $t \in \mathbf{R}$ and $p \in \text{ext } V_1^*$. Then again, as in the mentioned proof, we find $t = 0$ or 1.

REMARKS. Most of Theorem 3.4 and Corollary 3.7 are well known in the real case. The max + min characterization of Corollary 3.5 may look new, but is a consequence of [L–U, Theorem 3.8]. In the real case the contributors have been Effros [E], Fakhoury [F], Lima and Uttersrud [L–U], Lindenstrauss and Wulbert [L–W] and Ka-Sing Lau [LAU].

Complex C_σ-spaces were introduced by Olsen [O] and (i) ⇔ (vii) of Theorem 3.4 was first proved by him. Friedman and Russo [F–R_1], [F–R_2] work in the context of J^*-algebras. They have proved that a commutative J^*-algebra can be represented as a complex C_σ-space, and some of the results of Theorem 3.4 and Corollary 3.7 may be found partly in their papers. The rest of Theorem 3.4, Theorem 3.6 and Corollary 3.7 seems to be new.

ACKNOWLEDGEMENT

Lemma 2.3 is vital to this paper. A part of the proof consists of demonstrating that a certain semi M-ideal is already an M-ideal. I am grateful to Åsvald Lima for pointing out to me how results of his could be used to prove that. He has also read this manuscript and made valuable suggestions for improvement.

I would also like to thank the referee for his helpful comments and suggestions.

REFERENCES

[A–E] E. Alfsen and E. Effros, *Structure in real Banach spaces*, Ann. of Math. **96** (1972), 98–173.

[E] E. Effros, *On a class of real Banach spaces*, Isr. J. Math. **9** (1971), 430–458.

[F] H. Fakhoury, *Preduaux de L-espace, Notion de Centre*, J. Funct. Anal. **9** (1972), 189–207.

[F–R_1] Y. Friedman and B. Russo, *Contractive projections on $C_0(K)$*, Trans. Am. Math. Soc. **273** (1982), 57–73.

[F–R_2] Y. Friedman and B. Russo, *Function representation of commutative operator triple systems*, J. London Math. Soc. **27** (1983), 513–524.

[L_1] Å. Lima, *Intersection properties of balls and subspaces in Banach spaces*, Trans. Am. Math. Soc. **227** (1977), 1–62.

[L_2] Å. Lima, *Uniqueness of Hahn-Banach extensions and liftings of linear dependences*, Math. Scand. **53** (1983), 97–113.

[L–R] Å. Lima and A. Roy, *Characterizations of complex L_1-preduals*, Quart. J. Math. Oxford (2) **35** (1984), 439–453.

[L–U] Å. Lima and U. Uttersrud, *Centers of symmetry in finite intersections of balls in Banach spaces*, Isr. J. Math. **44** (1983), 189–200.

[L-W] J. Linaenstrauss and D. Wulbert, *On the classification of the Banach spaces whose duals are L_1-spaces*, J. Funct. Anal. **4** (1969), 332–349.

[LAU] Ka-Sing Lau, *The dual ball of a Lindenstrauss space*, Math. Scand. **33** (1973), 323–337.

[LIN] J. Lindenstrauss, *Extension of compact operators*, Mem. Am. Math. Soc. **48** (1964).

[LOU] Å. Lima, G. Olsen and U. Uttersrud, *Intersections of M-ideals and G-spaces*, Pac. J. Math. **104** (1983), 175–177.

[O] G. Olsen, *On the classification of complex Lindenstrauss spaces*, Math. Scand. **35** (1974), 237–258.

[RAO] T. Rao, *Characterizations of some classes of L_1-preduals by the Alfsen–Effros structure topology*, Isr. J. Math. **42** (1982), 20–32.

[ROY] N. Roy, *An M-ideal characterization of G-spaces*, Pac. J. Math. **92** (1981), 151–160.

[T] P. Taylor, *A characterization of G-spaces*, Isr. J. Math. **10** (1971), 131–134.

[U] U. Uttersrud, *On M-ideals and the Alfsen–Effros structure topology*, Math. Scand. **43** (1978), 369–381.